Constructive role of temperature in ratchets driven by trichotomous noise

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The dynamics of an overdamped Brownian particle in a piecewise linear spatially periodic potential subjected to both thermal and colored symmetric three-level Markovian (trichotomous) noises is investigated. In the case of large flatness, the exact formula for the stationary current is presented. The dependence of the current on the system parameters is analyzed and the conditions for the occurrence of current reversals are found. It is shown that the direction and value of the current can be controlled by a thermal noise. Asymptotic formulas for the current for various limits of the noise parameters are calculated and compared with the results of other authors. For small noise amplitudes, it is demonstrated that the temperature at which the current is maximized is proportional to the height of the potential barrier, being a slowly varying function of the other system parameters.

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I. INTRODUCTION

The past six years have witnessed an increasing interest in the study of noise-induced transport in spatially periodic structures called ratchets (for a reference survey, see [1–3]). It was argued in [4] that a ratchet (Brownian motor) could extract energy even from zero-mean value nonequilibrium fluctuations. The initial motivation in this field has come from cell biology, in particular from studies of the mechanism of vesicle transport inside eukariotic cells, via the motor proteins along microtubules [2,4–6]. Another motivation arises from the possible new methods of particle separation [7–9]. Later on, new systems with the same underlying ideas for the transportation were proposed, such as the recognition of the ''ratchet effect'' in the quantum domain [10–13].

Many different forms of ratchet systems are possible, since ratchet systems do not obey a detailed balance that can be violated in many different ways. The classification of different types of ratchets (correlation, flashing, etc.) is in Ref. [1]. Among them, we can mention the ''correlation ratchets,'' in which the particles move in a spatially periodic static potential driven by a nonthermal noisy force. The necessary condition for net movement in one direction is that the potential has no inversion symmetry or the fluctuations are statistically asymmetric in the sense that their odd-numbered higher-order cummulants are not identically zero [14].

It should be noted that the dynamics in ratchet structures with its inherent spatial asymmetry generally exhibits a rich complexity, such as the occurrence of multiple current reversals (CRs) and multipeaked current characteristics [1,2]. Also the combined influence of several different noise sources can cause unexpected behavior in the system [4,15–19]. Two noises acting together can potentially generate a far more organized motion than either of them alone, even though the noise sources are statistically independent [15]. The authors of Ref. [17] have analyzed a correlation ratchet, in which directed transport is subjected to both a thermal equilibrium noise and zero-mean asymmetric dichotomous fluctuations. They have shown that the transport direction of Brownian particles can be controlled by thermal noise, i.e., the presence of an additional thermal noise can cause CRs. Moreover, the dependence of the current on the temperature is nonmonotonic and there are two other characteristic (optimal) temperatures at which, respectively, the positive and negative currents are maximized.

The models with CRs are potentially very useful, because CRs could lead to a more efficient fluctuation-induced separation of particles [8,20–22]. This fact has partly motivated many works in which the CR phenomenon is also considered (for a reference survey, see [1,2] and also [23]). For example, it has been shown that the effect of CRs can be attained by changing the correlation time of nonequilibrium fluctuations as well as the flatness parameter (the ratio of the fourth moment to the square of the second moment) of the noise [20,24–26]. The direction of the current can also be reversed by modifying either the power spectrum of the noise source [27] or the number of interacting Brownian particles per unit cell [28], the mass of the particles [29], the temperature in multinoise cases [17], etc.

Nevertheless, most of the results have been obtained by numerical methods or for limits of slow and fast noises. There are not many exact results known for correlation ratchets, enabling us to quantitatively evaluate the values of the noise parameters corresponding to CRs for concrete models, or giving sufficient and necessary conditions for their existence [23,24,27,30]. This is caused, first and foremost, by the fact that even simple model ratchets display a rich variety of behaviors that vary remarkably with the system parameters. It would be quite difficult to capture the full range of these possibilities and the transitions between them at changing parameters by numerical solutions alone.

Unfortunately, the multinoise case is difficult to treat analytically. However, the advantage of multinoise models with

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a thermal noise is that the control parameter is temperature, which can easily be varied in experiments as well as potential technological applications.

In this paper, we consider one-dimensional overdamped dynamical systems determined by a first-order differential equation with a periodic piecewise linear potential and with an additive noise term composed of a trichotomous and some thermal noise. The trichotomous process is a symmetric three-level stationary telegraph process characterized by three parameters: amplitude $a_0 \in (0,\infty)$, correlation time $\tau_c \in (0,\infty)$, and flatness $\varphi \in (1,\infty)$ [23,31,32]. The purpose of this paper is to provide exact analytical results for the stationary current J over extended trichotomous noise parameters and temperature regimes of the system. We show that the direction and value of the current can be controlled by temperature. In the case of a large flatness, we have succeeded in reaching conditions which bring forth CRs.

The structure of the paper is as follows. Section II presents the model and exact differential equation for the stationary probability density. Section III analyzes the behavior of the current at different limits, such as the zero-temperature case, slow-noise limit, large-amplitude limit, etc. Section IV focuses on the case of a large flatness. The stationary current is found and the dependence of CRs on the noise parameters and the temperature is investigated. In Sec. V, the physically important case of a small noise amplitude is analyzed and the interval of temperature that maximizes the current is estimated. Section VI contains some concluding remarks.

II. MODEL WITH A TRICHOTOMOUS MARKOVIAN NOISE

Let us consider an overdamped motion of Brownian particles in a one-dimensional spatially periodic potential $\tilde{V}(\tilde{x}) = \tilde{V}(\tilde{x}+L)$ of a period L and a barrier height $\tilde{V}_0 = \tilde{V}_{\max} - \tilde{V}_{\min}$. Its dynamics is determined by the stochastic differential equation

$$\kappa \frac{d\tilde{X}}{d\tilde{t}} = \tilde{h}(\tilde{X}) + \tilde{Z}(\tilde{t}) + \tilde{\xi}(\tilde{t}), \tag{1}$$

where $\tilde{h}(\tilde{x}) = -(d/d\tilde{x})\tilde{V}(\tilde{x})$ is the deterministic force.

The thermal fluctuations $\tilde{\xi}(\tilde{t})$ are modeled by a zero-mean Gaussian white noise with the correlation function $\langle \tilde{\xi}(\tilde{t}_1), \tilde{\xi}(\tilde{t}_2) \rangle = 2\kappa k_B T \delta(\tilde{t}_1 - \tilde{t}_2)$, where κ is the friction coefficient, k_B is the Boltzmann constant, and T is the temperature of the system. The random force $\tilde{Z}(\tilde{t})$ represents nonequilibrium fluctuations assumed to be a zero-mean trichotomous Markovian stochastic process [23,31,32] taking the values $\tilde{z} = \{\tilde{a}_0, 0, -\tilde{a}_0\}, \tilde{a}_0 > 0$. The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities $P_s(\tilde{a}_0) = P_s(-\tilde{a}_0) = q$, $P_s(0) = (1-2q)$. The marginal density $p(\tilde{z},\tilde{t})$ for this process evolves according to

$$\frac{\partial p(\tilde{z},\tilde{t})}{\partial \tilde{t}} = -\tilde{\nu}\left(p(\tilde{z},\tilde{t}) - P_s(\tilde{z}) \int p(\tilde{z}',\tilde{t}) d\tilde{z}' \right),$$

so that the trichotomous process is a particular case of the kangaroo process [24], with a correlation time $\tilde{\tau}_c = 1/\tilde{\nu}$ and with the flatness parameter $\varphi = \langle \tilde{Z}^4(\tilde{t}) \rangle / \langle \tilde{Z}^2(\tilde{t}) \rangle^2 = 1/(2q)$. In a stationary state, the fluctuation process $\tilde{Z}(\tilde{t})$ satisfies $\langle \tilde{Z}(\tilde{t}+\tilde{\tau}), \tilde{Z}(\tilde{t}) \rangle = 2q\tilde{a}_0^2 \exp(-\tilde{\nu}\tilde{\tau})$ and the noise intensity is $\tilde{\sigma}^2 = 4q\tilde{a}_0^2/\tilde{\nu}$, i.e., it is a symmetric zero-mean exponentially correlated noise.

By applying a scaling of the form

$$X = \frac{\tilde{X}}{L}, \quad t = \frac{\tilde{t}}{t_0}, \quad Z = \frac{L\tilde{Z}}{\tilde{V}_0}, \quad V(x) = \frac{\tilde{V}(\tilde{x})}{\tilde{V}_0}, \quad \xi = \frac{L\tilde{\xi}}{\tilde{V}_0} \tag{2}$$

we get a dimensionless formulation of the dynamics with the potential V with the property $V(x) = V(x-1)$. By the choice $t_0 = \kappa L^2/\tilde{V}_0$, the dimensionless friction coefficient turns to unity. The rescaled noise is given by

$$\nu = \frac{\kappa L^2 \tilde{\nu}}{\tilde{V}_0}, \quad a_0 = \frac{L\tilde{a}_0}{\tilde{V}_0}, \quad D = \frac{k_B T}{\tilde{V}_0}, \tag{3}$$

where $2D$ is the strength of the rescaled zero-mean Gaussian white noise $\xi(t)$. For brevity's sake, from now on we shall call D temperature. The dynamics reads

$$\frac{dX}{dt} = h(X) + Z(t) + \xi(t), \quad h(x) \equiv -\frac{dV(x)}{dx}. \tag{4}$$

The joint probability density for the position variable $x(t)$ and the fluctuation variable $z(t)$, $P_n(x,t)$, satisfies the Fokker-Planck master equation

$$\frac{\partial}{\partial t} P_n(x,t) = -\frac{\partial}{\partial x}\left[\left(h(x) + z_n - D\frac{\partial}{\partial x} \right) P_n(x,t) \right] + \sum_m U_{nm} P_m(x,t), \tag{5}$$

with $P_n(x,t)$ denoting the probability density for the combined process (x, z_n, t); $n,m = 1,2,3$; $z_1 \equiv -a_0$, $z_2 \equiv 0$, $z_3 \equiv a_0$, and

$$\mathbf{U} = \nu \begin{pmatrix} q-1 & q & q \\ 1-2q & -2q & 1-2q \\ q & q & q-1 \end{pmatrix}. \tag{6}$$

The stationary current J is then evaluated via the current densities

$$j_n(x)=\left(h(x)+z_n-D\frac{d}{dx}\right)P_n^s(x),$$

$$J=\sum_n j_n(x), \tag{7}$$

where $P_n^s(x)$ is the stationary probability density for the state (x,z_n). It follows from Eq. (5) that the current J is constant.

For the calculation of the stationary probability density in the x space, $P(x)=\Sigma_n P_n^s(x)$, and the stationary current $J=\text{const}$, the following differential equation can be obtained from Eq. (5):

$$\begin{aligned}\nu\psi(x)&+\frac{d}{dx}\left(h(x)\psi(x)-2qa_0^2P(x)-D\frac{d}{dx}\psi(x)\right)\\&+\frac{d}{dx}\left\{\frac{1}{\nu+h'(x)}\left(h(x)-D\frac{d}{dx}\right)\left[\nu\psi(x)\right.\right.\\&\left.\left.+\frac{d}{dx}\left(h(x)\psi(x)-2qa_0^2P(x)-D\frac{d}{dx}\psi(x)\right)\right]\right\}\\&=(1-2q)a_0^2\frac{d}{dx}\left[\frac{1}{\nu+h'(x)}\frac{d}{dx}\psi(x)\right],\end{aligned} \tag{8}$$

where

$$h'(x)\equiv\frac{d}{dx}h(x),\quad \psi(x)\equiv-J+h(x)P(x)-D\frac{d}{dx}P(x). \tag{9}$$

This is a (nonautonomous linear) fifth-order ordinary differential equation with, additionally, the probability current J to be determined. Two conditions are imposed on it: periodicity $P(x)=P(x+1)$ and normalization of $P(x)$ over the period interval $L=1$ of the rescaled ratchet potential $V(x)$:

$$\int_0^1 P(x)dx=1. \tag{10}$$

These two conditions are sufficient for a unique solution of Eq. (8). The combination of Eqs. (8)–(10) with Eq. (4) yields the following relation between the average of the particle velocity $\langle dX/dt\rangle$ and the current J:

$$\langle dX/dt\rangle=\langle h(X)\rangle=\int_0^1 h(x)P(x)dx=J. \tag{11}$$

In the case of $q=\frac{1}{2}$ (a dichotomous noise), the last term in Eq. (8) vanishes and Eq. (8) is satisfied by every solution of the equation

$$\nu\psi(x)+\frac{d}{dx}\left(h(x)\psi(x)-a_0^2P(x)-D\frac{d}{dx}\psi(x)\right)=0.$$

The latter corresponds to Eq. (4) in case $Z(t)$ is a dichotomous noise. This has been investigated by several authors [15–17,24].

Unfortunately, exact solutions of Eq. (8) can be obtained in only a few cases. Here we consider a piecewise linear (sawtoothlike) potential, the dimensionless form of which reads

$$V(x)=\begin{cases}-(x-d)/d, & x\in(0,d)\ \text{mod}\ 1,\\(x-d)/(1-d), & x\in(d,1)\ \text{mod}\ 1,\end{cases} \tag{12}$$

where $d\in(0,1)$ determines the asymmetry of the potential, which is symmetric if $d=\frac{1}{2}$. We may confine ourselves to the case $d\leqslant\frac{1}{2}$. As our starting equation (8) has been derived at the assumption that $V(x)$ is differentiable at every point, we have to consider the sawtooth potential as a limit case of a smooth potential, so that

$$h(d+k)=h(k)=0,$$

with k being an integer. The force $h(x)$ being periodic, the stationary distribution $P(x)$ as a solution of the problem Eqs. (8)–(10) is also periodic and it suffices to consider the problem in the interval $[0,1)$. The force corresponding to the potential (12) is

$$h(x)=-\frac{dV(x)}{dx}=\begin{cases}b:=1/d, & x\in(0,d),\\-c:=-1/(1-d), & x\in(d,1).\end{cases} \tag{13}$$

As the force $h(x)$ is a piecewise constant in the open intervals $(0,d)(\text{mod}\,1)$ and $(d,1)(\text{mod}\,1)$ with a discontinuity at points $x_1=d(\text{mod}\,1)$, $x_2=1(\text{mod}\,1)$, Eq. (8) splits up into two fifth-order linear differential equations with constant coefficients for two functions $P_i(x)$ $(i=1,2)$ defined on the intervals $(0,d)$ and $(d,1)$, respectively. The solution reads

$$P_i(x)=\frac{J}{h_i}+\sum_{k=1}^{5}C_{ik}e^{\lambda_{ik}x/D}, \tag{14}$$

where $h_1:=b$, $h_2:=-c$, C_{ik} are constants of integration, and $\{\lambda_{ik},k=1,\ldots,5\}$ is the set of roots of the algebraic equation

$$\begin{aligned}\lambda_i^5&-3h_i\lambda_i^4+(3h_i^2-a_0^2-2\nu D)\lambda_i^3+(4D\nu+a_0^2-h_i^2)h_i\lambda_i^2\\&+\nu D(\nu D-2h_i^2+2qa_0^2)\lambda_i-D^2\nu^2h_i=0.\end{aligned} \tag{15}$$

Ten conditions, at the points of discontinuity, follow from Eqs. (5) and (8):

$$P_1(x_k)=P_2(\overline{x_k}),\quad \psi_1(x_k)=\psi_2(\overline{x_k}),$$

$$\hat{T}_1\psi_1(x_k)=\hat{T}_2\psi_2(\overline{x_k}),$$

$$\hat{T}_2^2\psi_2(\overline{x_k})-\hat{T}_1^2\psi_1(x_k)=(b+c)C_k,$$

$$\frac{d}{dx}(\hat{T}_1^2-a_0^2)\psi_1(x_k)=\frac{d}{dx}(\hat{T}_2^2-a_0^2)\psi_2(\overline{x_k}),$$

where $k=1,2$, $\hat{T}_i:=h_i-D(d/dx)$, $\psi_i(x):=\hat{T}_iP_i(x)-J$, $x_1=\overline{x_1}=d$, $x_2=0$, $\overline{x_2}=1$, $C_1:=C_2+\nu\int_0^d\psi(x)dx$, and the constant C_2 is defined by

$$C_2 = 2qa_0^2 - \int_0^1 h(x)\psi(x)dx - \nu\int_0^1 dx'\int_0^{x'}\psi(x)dx.$$

By including the eleventh (normalization) condition, one obtains a complete set of conditions for the ten constants of integration of Eq. (14) and for the probability current J. This procedure leads to an inhomogeneous set of eleven linear algebraic equations. Now, J can be expressed as a quotient of two determinants of the eleventh degree. However, this complex formula is not reproduced here and instead the exact results are analyzed for their corresponding different asymptotic regimes.

III. ASYMPTOTIC REGIMES

A. The zero-temperature case

The case of zero temperature has been considered in detail in [23]. Here is a short review of the points needed further on.

The following characteristic regions can be discerned for the noise amplitude a_0.

(i) There is no current, if $0<a_0<c$, as there is a stationary stable point for any state n.

(ii) In the case of $c<a_0<b$, there exists one stationary stable point for $z(t)=-a_0$, the motion to the left is switched off, and the current is positive.

(iii) In the case of $a_0>b$, the stochastic process $Z(t)$ can, though it should not, induce a reversal of the current. Now we shall consider this case in some detail.

In the phase space of the parameters φ, a_0 one can distinguish between four domains of qualitatively different shapes of the current $J(\nu)$, characterized also by sign reversals. Three circumstances should be pointed out: (i) there is a lower limit for the noise amplitude, namely $a_0=b+c$, below which there is no CR at any τ_c or φ; (ii) the correlation time $\tau_c=1/\nu$ has an upper limit over which there cannot be more than one CR; (iii) the flatness parameter φ has a critical value $\varphi=2$. If $\varphi<2$, then, as the correlation time grows from 0 to ∞, there can be either two reversals or none, and if $\varphi>2$, one reversal can but need not occur. It is remarkable that at sufficiently large noise amplitudes, $a_0^2 \gg \max\{b^2/q^2(1-2q),\ \nu/2q(1-2q)\}$, the behavior of the current is completely due to the effect of the flashing barrier for all values of the correlation time and the flatness parameter. In the case of fixed values of φ and τ_c, the current saturates to a finite negative value at great noise amplitudes:

$$J = -\frac{1-2q}{2q\nu}[b^2(1-e^{-2\nu q/b^2}) - c^2(1-e^{-2\nu q/c^2})]. \tag{16}$$

In the fast-noise limit ($\nu\to\infty$, all other parameters fixed), the current is transcendentally small. In this case, the current is positive if $\varphi<2$ and negative if $\varphi>2$.

B. The large-amplitude limit

For $a_0\to\infty$ and for fixed ν, q, and D, i.e., for the case of a very large noise intensity $\sigma^2=4qa_0^2/\nu\gg1$, the current saturates at the value

$$J = -\frac{(1-2q)b^2c^2\{\eta(1-\alpha^2)(\varepsilon-\beta)(1-\varepsilon\beta) - \gamma[(1-\beta^2)(\varepsilon-\alpha)(1-\varepsilon\alpha) + \eta(1-\varepsilon^2)(1-\alpha\beta)(\alpha-\beta)]\}}{2\varepsilon\nu q\{\gamma\eta[(1-\alpha\beta)^2+(\alpha-\beta)^2] + (8q\nu D/bc-1)[(1-\alpha\beta)^2-(\alpha-\beta)^2]\}}, \tag{17}$$

where $\gamma := \sqrt{1+8\nu qD/b^2}$, $\eta := \sqrt{1+8\nu qD/c^2}$, $\varepsilon := \exp(-1/2D)$, $\alpha := \exp(-\gamma/2D)$, $\beta := \exp(-\eta/2D)$.

It should be noted that in this case the current is negative (or zero) for all finite values of ν, D, d, and q. Obviously, if $q=\frac{1}{2}$, i.e., for dichotomous noises, or if the potential is symmetric ($b=c$), there is no current in the stationary state. Equation (17) reveals a new quantitative aspect of the interaction of the noises: they combine to produce a new length scale $\sqrt{D/\nu}$ (see also [15]). This length scale is a typical distance the particle can diffuse between switches of the trichotomous force. There is no such length scale in the absence of one of the fluctuation sources. At limit of zero temperature, we find that the current is given by Eq. (16). If $D\to\infty$, we have

$$J \approx -(1-2q)\frac{\nu q(b-c)}{180bcD^3}, \tag{18}$$

thus the current decays algebraically to zero in D^{-3} at a rate proportional to τ_c^{-1}.

In the case of $a_0^2 \gg q\nu D\to\infty$, while D is finite, we obtain

$$J \approx -(1-2q)\frac{2(b-c)b^2c^2D}{(8q\nu D)^{3/2}} \tag{19}$$

and the current vanishes algebraically in $(q\nu D)^{-3/2}$ at a rate proportional to D.

The condition $q\nu D\gg1$ takes a distinct physical meaning: the length scale $\sqrt{D/\nu}$ is much larger than the typical distance the particle is driven by the deterministic force in the state $z=0$ of the trichotomous noise.

In the asymptotic limit of $q\nu\to0$, i.e., if the trichotomous noise changes slowly, J is given by

$$J \approx -\frac{(1-2q)\nu q(b-c)}{bc}f(D), \tag{20}$$

where

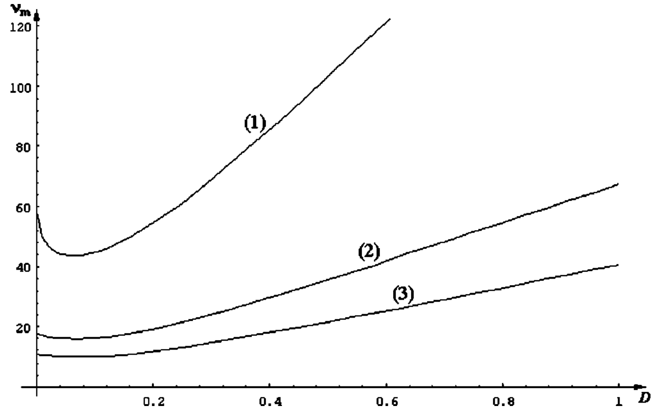

FIG. 1. The switching rate ν_m that minimizes the current [see Eq. (17)] vs the dimensionless temperature D. The curves (1)–(3) correspond to the following parameters: (1) $q=0.25$, $d=0.05$; (2) $q=0.25$, $d=0.25$; (3) $q=\frac{1}{3}$, $d=0.45$. If $D>1$, then ν_m is nearly proportional to the temperature.

$$f(D) := 1-4D+\frac{2}{D(e^{1/D}-1)}\left(D+\frac{e^{1/D}}{e^{1/D}-1}\right). \qquad (21)$$

As D increases, the function $f(D)$ monotonically decreases from 1 to 0.

We can see that the current J tends to zero as $\nu q\to\infty$ or as $\nu q\to 0$. Consequently, J reaches a minimum as a function of ν. The value of the correlation time that minimizes the current ($\tau_{cm}=1/\nu_m$), being a solution of a transcendental equation, can in a general case be found by numerical calculation. Some of its properties can be analyzed analytically, though. As the temperature grows from zero to infinity, the parameter ν_m starts from a finite value $\nu_m(0)$, decreasing to a minimum, and then grows monotonically (see Fig. 1).

For $D>1$, the following approximate equation seems acceptable:

$$\nu_m \approx \frac{3bc}{2q}D\left(1+\frac{11\sqrt{3}}{10}\sqrt{\frac{c}{b}}\right), \qquad (22)$$

so that the parameter ν_m is proportional to the temperature and to the flatness parameter.

According to numerical calculations by various values of the system parameters, the application of Eq. (22) does not cause an error exceeding a few percent.

At large values of the potential asymmetry, $b\gg c$, and if the condition $D\ll 1$ is also fulfilled, ν_m can be given as

$$\nu_m \approx \nu_m(0)\left(1-\frac{2(b-c)D}{c}\right). \qquad (23)$$

Hence, ν_m decreases if the temperature increases. The correlation time $\tau_{c0}=1/\nu_m(0)$, which minimizes the current in the case of zero temperature, can be found by the following transcendental equation [where $x=2q\nu_m(0)$]:

$$(x+b^2)e^{-x/b^2}=(b^2-c^2)+(x+c^2)e^{-x/c^2}. \qquad (24)$$

If $b\gg c$, we have

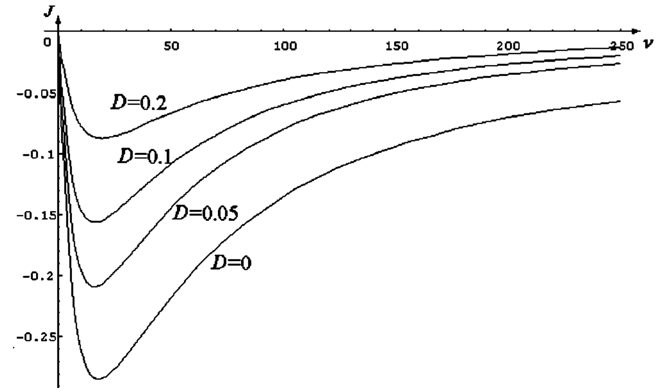

FIG. 2. The current J vs the switching rate ν at different temperatures D in the case of the large-amplitude limit [Eq. (17)]. The flatness parameter equals $\varphi=1/2q=2$ and the potential asymmetry parameter is $d=0.25$. The current is negative and its absolute value decreases monotonically as D increases.

$$\nu_m(0)\approx\frac{bc}{\sqrt{2}q}\left(1+\frac{\sqrt{2}c}{3b}+\frac{11c^2}{36b^2}\right).$$

The dependence of the current on the temperature and the switching rate ν is illustrated in Fig. 2. As the temperature grows, the current decreases monotonically to zero at any values of the parameters q,ν, and d.

It is remarkable that in the case of fixed values of φ,τ_c, and D, the current saturates to a finite value at large noise amplitudes ($a_0\to\infty$). This somewhat surprising result is due to both an effective inhomogeneous diffusion, which becomes more homogeneous at an increasing a_0, and the so-called flashing barrier effect as stated in Refs. [20,23].

C. The white-noise limit

In the trichotomous δ-correlated limit, $\nu\to\infty, a_0\to\infty$, so that $\sigma^2=4qa_0^2/\nu$ is finite, the solution of problem (8) reduces to

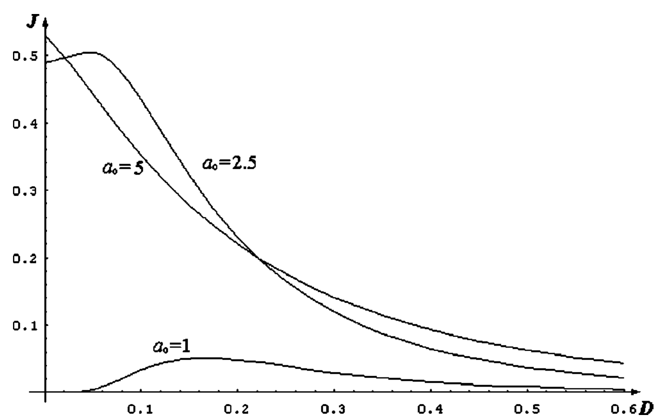

FIG. 3. The current J vs the temperature D at various noise amplitudes a_0 in the case of an adiabatic limit [Eq. (26)]. The flatness parameter equals $\varphi=1.5$ and the potential asymmetry parameter is $d=0.25$. Note that for $a_0<b=1/d$, the current exhibits a bell-shaped extremum; if $a_0>b$, then J decreases monotonically. The temperature that maximizes the current decreases monotonically as a_0 increases.

$$J \approx \frac{\sigma^4 (b^2 - c^2) e^{1/D^*}}{4\nu D^{*5} (e^{1/D^*} - 1)^2} (2 - \varphi), \tag{25}$$

where

$$D^* := \frac{\sigma^2}{2} + D, \quad \varphi := \frac{1}{2q}.$$

The current in this limit is proportional to the noise correlation time that in this case is a measure of the distance from equilibrium. The current in Eq. (25) has a factor dependent on noise statistics via the flatness parameter φ.

As the flatness parameter grows, a current reversal appears at $\varphi = 2$ in complete accordance with the results of [24], where the general kangaroo process is considered. The current J takes an extremum at the effective temperature $D^* \approx 0.203$. Hence, if the intensity of the trichotomous noise σ^2 is less than 0.406, there is a characteristic temperature D_m that maximizes (for $\varphi > 2$ minimizes) the current: $D_m = 0.203 - \sigma^2/2$. If the intensity σ^2 exceeds the critical value 0.406, J decreases monotonically to zero as the temperature increases.

D. The adiabatic limit

At the long-correlation-time limit $\nu \to 0$, the current saturates at the value

$$J = J_0(b,c) - J_0(c,b), \tag{26}$$

where

$$J_0(b,c) = \frac{q(a_0+b)^2 (a_0-c)^2}{(a_0+b)(a_0-c)(b-c+a_0) - D(b+c)^2 (e^{(a_0+b)/bD} - 1)(e^{(a_0-c)/cD} - 1)/(e^{a_0/D} - 1)}. \tag{27}$$

The form of the current J essentially coincides with that in [4] in the case of a dichotomous noise. For the adiabatic limit the current is positive and changes with temperature as follows: in case the trichotomous fluctuations induce transitions back and forth over the potential barrier, i.e., if $a_0 > b$, J decreases monotonically as the temperature increases. On the other hand, if the trichotomous transitions do not induce transitions in both directions over the barrier, i.e., if $a_0 < b$, the net current exhibits a bell-shaped extremum (see Fig. 3). Hence, if $a_0 < b$, there is an optimal temperature D_m maximizing the current. As the noise amplitude a_0 increases, the temperature D_m decreases monotonically to zero at $a_0 = b$.

For small noise amplitudes, $a_0 \ll \min\{c, D\}$, one finds from Eqs. (26) and (27) that

$$J \approx \frac{q a_0^2 (b-c) e^{1/D}}{D^3 bc (e^{1/D} - 1)^2} f(D), \tag{28}$$

where $f(D)$ has been defined in Eq. (21). It is easy to ascertain that the optimal temperature $D_m \approx 0.216$. It is remarkable that in this case the characteristic temperature D_m depends neither on the shape of the ratchet potential nor on the parameters of the trichotomous noise. It seems reasonable to assume that for overdamped ratchet models with an additive thermal noise and with an additive low-amplitude nonequilibrium noise, the same value of the optimal temperature in the adiabatic limit occurs.

E. The fast-noise limit

In the fast-noise limit, we allow ν to become large, holding all other parameters fixed, and use $\nu^{-1/2}$ as a smallness parameter. Thus, if $D \neq 0$, in the large ν limit the current can be given as

$$J \approx \frac{q a_0^2 bc (b^2 - c^2) e^{1/D}}{2\nu^{5/2} D^{7/2} (e^{1/D} - 1)^2}, \tag{29}$$

so that the current is positive and decays algebraically to zero in $\nu^{-5/2}$. In the case of a dichotomous noise ($q = \frac{1}{2}$), such a formula for J has been found in [15]. The thermal noise has a strong effect on the current in the small τ_c limit: in the presence of thermal noise fluctuations, the current takes exponential growth from $J \sim \pm \exp(-C/\tau_c)$ with a positive constant C that depends on a_0, q, and d, to $J \sim \tau_c^{5/2}$. It looks like in this model the two noises acting together are able to generate a considerably more organized motion than either one of them alone, even though they are generated by statistically independent sources. The authors of Ref. [15] have reached an analogous conclusion for the case of a dichotomous noise. Extreme sensitivity to thermal noise can be seen from the factor $e^{1/D}/(e^{1/D} - 1)^2$ in Eq. (29) that decays exponentially at a small D and the current drops like $O(e^{-1/D})$ as $D \to 0$. Notably, the limits $D \to 0$ and $\nu \to \infty$ do not commute in the formulas for the current J (see also [16]). It should also be noted that though in the case of $\varphi > 2$ and $D = 0$ there can also occur one CR caused by variation of ν, in the case of $\varphi > 2$ and $D \neq 0$ there can occur either two reversals or none.

It can be seen easily that the functional dependence of the current on the temperature D is of a bell-shaped form. The optimal temperature D_m that maximizes the current equals 0.309.

IV. TRANSPORT IN THE CASE OF LARGE FLATNESS

A. The large-flatness limit

At a large-flatness parameter, i.e., $\varphi \gg 1$, a natural way to investigate the behavior of J is to apply small-q perturbation expansions. The stationary solution of Eq. (8) with $D \neq 0$ is constructed in terms of integer powers of q. The current can be expressed as $J=qJ^{(1)}+q^2J^{(2)}+\cdots$. We shall calculate the leading term of the current $qJ^{(1)}$. Notably, the analysis of this section is valid for the values of parameters satisfying the conditions

$$|\Delta\rho_{ik}|\equiv|\rho_{ik}-\rho_{ik}^0|\ll\min\{|h_i|,|\rho_{ik}^0|\},\quad i=1,2, \tag{30}$$

where $\{\rho_{ik},k=1,\ldots,5\}$ is the set of roots of the algebraic equation (15) and $\{\rho_{ik}^0,k=1,\ldots,5\}$ is the set of roots of Eq. (15) for $q=0$. These conditions result from the assumption that the higher-order terms in the expansion of the roots of Eq. (15) are asymptotically smaller than the lower-order terms held in the calculation, and that the exponential terms $\exp(\Delta\rho_{ik}/h_i)$ of the current J can be linearized, i.e., $\exp(\Delta\rho_{ik}/h_i)\approx1+\Delta\rho_{ik}/h_i$.

The exact formula for the leading-order term $qJ^{(1)}$ of the current is

$$qJ^{(1)}=\frac{q\nu a_0^2b^2c^2e^{1/D}}{(e^{1/D}-1)^2}\Bigg[\tilde{J}(b,c)-\tilde{J}(c,b)-\frac{2a_0^2(b-c)}{Dbc(\nu^2D^2-c^2a_0^2)(\nu^2D^2-b^2a_0^2)}\Bigg], \tag{31}$$

where

$$\tilde{J}(b,c)=\frac{(\nu D+a_0^2)\{\gamma_1\gamma_2[\alpha_1\beta_1(\alpha_2+\beta_2)-\beta_2\alpha_2(\alpha_1+\beta_1)]+\delta(a_0+b)+\tilde{\delta}(a_0-c)\}+2\nu D[\delta(a_0-b)+\tilde{\delta}(a_0+c)]}{(\nu D-ca_0)^2(\nu D+ba_0)^2\{\gamma_1\gamma_2[\varepsilon_1\varepsilon_2+2(\alpha_1-\alpha_2)(\beta_2-\beta_1)]+\varepsilon_1\varepsilon_2[4\nu D+(a_0+b)(a_0-c)]\}} \tag{32}$$

and

$$\gamma_i:=\sqrt{4\nu D+(a_0+h_i)^2},\quad h_1:=b,\quad h_2:=-c,$$

$$\varepsilon_i:=1-\exp\left(-\frac{\gamma_i}{D|h_i|}\right),$$

$$\alpha_i:=1-\exp\left[-\frac{1}{2D}\left(\frac{a_0-h_i}{h_i}+\frac{\gamma_i}{|h_i|}\right)\right],$$

$$\beta_i:=\exp\left[\frac{1}{2D}\left(\frac{a_0-h_i}{h_i}-\frac{\gamma_i}{|h_i|}\right)\right]-1, \tag{33}$$

$$\delta:=\varepsilon_1\beta_2\alpha_2\gamma_2,\quad \tilde{\delta}:=\varepsilon_2\beta_1\alpha_1\gamma_1.$$

The result (31)–(33) is not confined to the large-flatness model. Instead, it applies to a large class of asymptotic regimes. It is also valid for all asymptotic regimes in which the leading term is proportional to q. These are, for example, the adiabatic limit ($\nu\to0$), the fast-noise limit ($\nu\to\infty$), and the case of a small noise amplitude ($a_0\ll c$). It is not difficult to see that at the limits $\nu\to0$ and $\nu\to\infty$, Eqs. (31)–(33) reduce to Eqs. (26) and (29), respectively.

The exact formula (31) for $qJ^{(1)}$ is complex and as such not lucid enough. To get more information, we shall study it in the asymptotic limits. To visualize the exact results, computer graphics will be applied.

B. Asymptotics

(i) In the trichotomous δ-correlated limit, we have

$$J\approx qJ^{(1)}\approx\frac{-a_0^2\sigma^2(b^2-c^2)e^{1/D}}{2\nu^2D^5(e^{1/D}-1)^2}. \tag{34}$$

This result coincides with Eq. (25) if $\sigma^2/2\ll\min\{D,D^2\}$ and $\varphi\gg1$. The first condition, in which the intensity of the trichotomous noise is much smaller than that of the thermal noise, is in complete accordance with the conditions (30).

(ii) In the case of low temperature, $D\to0$, the conditions (30) reduce to $q\ll\min\{c/a_0,c^2/2\nu\}$. For $a_0>b$, the current behaves asymptotically as

$$J\approx\nu q\{[e^{\nu/c(a_0-c)}-e^{-\nu/b(a_0+b)}]^{-1}-[e^{\nu/b(a_0-b)}-e^{-\nu/c(a_0+c)}]^{-1}\}. \tag{35}$$

Notably, J is positive in the case of $a_0\leqslant bc$ at any ν. If the noise amplitude exceeds bc, then the current reverses to negative at $\nu=\nu_0$. The point of reversal ν_0, being a solution of the transcendental equation $J(\nu_0)=0$, can in a general case be found by numerical calculation. As the noise amplitude grows, the parameter ν_0 decreases in the region $a_0>bc$ monotonically from infinity to zero. If $a_0\gg bc$, the asymptotic formula $\nu_0\approx2b^2c^2/a_0^2$ can be of use.

For $c<a_0<b$, Eq. (31) takes the form

$$J\approx\nu q[e^{\nu/c(a_0-c)}-e^{-\nu/b(a_0+b)}]^{-1},$$

so that the current is positive for all values of ν and d. If $a_0<c$, then J vanishes like $O(e^{-C/D})$ with a positive constant C as $D\to0$.

(iii) In the asymptotic limit of high temperature, $D\to\infty$, we find that J is positive and decays algebraically to zero in D^{-4} at a rate proportional to a_0^2,

$$J\approx\frac{q(b-c)a_0^2}{180bcD^4}. \tag{36}$$

We note that this formula is valid for all values of the parameter q. To leading order of large D, this coincides with a result of [16] for the dichotomous case, $q=\frac{1}{2}$.

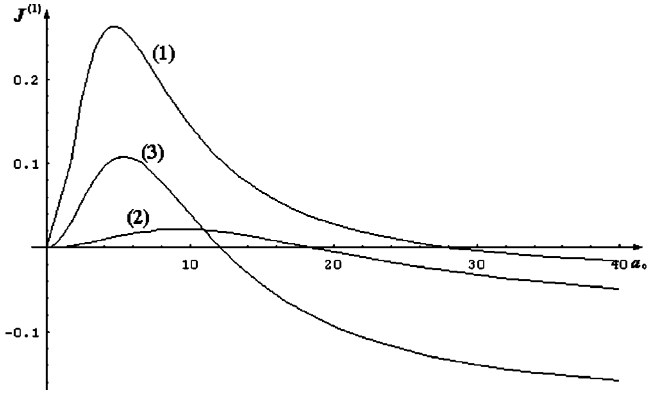

FIG. 4. The current $J=qJ^{(1)}$ vs the trichotomous noise amplitude a_0 at the potential asymmetry parameter $d=0.25$ in the case of large flatness [Eq. (31)]. Curves (1)–(3) correspond to the following parameters: (1) $\nu=1$, $D=0.4$; (2) $\nu=30$, $D=1$; (3) $\nu=10$, $D=0.5$. For large a_0, the current saturates at a finite negative value determined by Eq. (37).

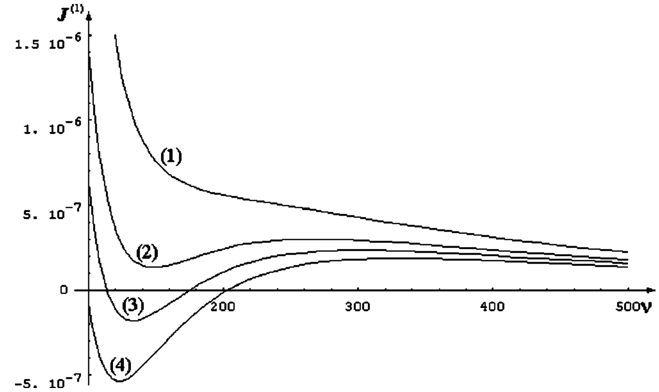

FIG. 5. The current $J=qJ^{(1)}$ vs the switching rate ν in the case of large flatness [Eq. (31)]. The curves have been computed for the noise amplitude $a_0=0.94$, the potential asymmetry parameter $d=0.25$, and temperatures: (1)$D=0.1$, (2)$D=0.098$, (3)$D=0.097$, (4)$D=0.096$. If $D<D_c(a_0)\approx 0.0976$ [curves (3) and (4)], two current reversals occur. The current J grows monotonically to a finite positive value as ν drops from 100 to zero.

(iv) For the large-amplitude limit, $a_0\to\infty$, the current saturates at the negative value

$$J=-\frac{\nu q(b-c)}{bc}f(D), \tag{37}$$

where the function $f(D)$ is defined by Eq. (21). This is valid for $2q\ll 1$ and $q\nu\ll\min\{c^2,c^2/D\}$ [see also Eq. (20)].

(v) For small amplitudes, $a_0\ll c$, the leading term of the current is positive and exhibits a bell-shaped form as D is varied. The behavior of J in this case will be considered in Sec. V.

Drawing on the asymptotic expressions of $J^{(1)}$, we can reach the following results. (i) For the variations of the amplitude a_0, an odd number of CRs occurs. (ii) At the variations of the correlation time τ_c, the number of CRs is even or zero. (iii) As for changes of the temperature D, we have to differentiate between two cases. First, if $a_0<bc$, or if $a_0>bc$ and $\tau_c>1/\nu_0$, there can occur either zero or an even number of CRs. Second, in the case of $a_0>bc$ and $\tau_c<1/\nu_0$, there is always an odd number of CRs. Moreover, in numerical calculations, varying the parameter D, we have not observed more than two subsequent CRs. Thus, for (iii) the possible number of CRs is zero, one, or two.

C. Current versus noise amplitude

We may look at the solution (31) as a function of a_0. At numerical analyses of the function $J^{(1)}(a_0)$ we have observed up to three CRs. For example, at the parameter values $d=0.005$, $D=0.02$, and $\nu=1000$, the current takes three zeros: $a_{01}\approx 15.79$, $a_{02}\approx 18.25$, and $a_{03}\approx 162.31$.

Still, in most cases there is only one CR. The typical form of the graph of $J(a_0)$ is represented in Fig. 4. The current has a positive maximum at a certain finite value a_m of a_0, and exhibits a reversal of the direction at $a_0=a^*>a_m$. For a large a_0, the current saturates at a finite negative value $J(\infty)$ [see also Eq. (37)]. At increasing temperature, the current $J(\infty)$ approaches zero monotonically. Following from Eq. (37), one might be tempted to postulate that for decreasing values of the correlation time τ_c, the absolute value of $J(\infty)$ increases monotonically. However, this occurs only for the values of $\nu\ll\min\{c^2/q,c^2/Dq\}$ where Eq. (37) is valid.

Though, in general, the parameters a_m and a^* cannot be expressed by elementary functions, at certain constraints rather simple approximate solutions can be found for them. For example, if $d>0.01, D>1/\rho$, and $\nu<15\rho D$, then the value of the noise amplitude a_m that maximizes the current is proportional to the temperature:

$$a_m\approx\rho bcD, \tag{38}$$

where ρ is the solution of the transcendental equation

$$2(b-c)=\rho^3\frac{d}{d\rho}\left\{\frac{1}{\rho^3(e^{\rho bc}-1)}[\rho bc(e^{\rho b}-e^{\rho c}) -2(b-c)(e^{\rho b}-1)(e^{\rho c}-1)]\right\}.$$

The relative error at the application of Eq. (38) does not exceed 10%.

D. Current versus switching rate

We may also look at the solution (31) as a function of the switching rate ν. The typical forms of the graph of $J(\nu)$ are represented in Fig. 5. There is a lower limit for the noise amplitude $a_1(D)$, which depends on the temperature below which there is no CR at any ν. Direct numerical calculations with various values of the system parameters indicate that at a moderate asymmetry of the potential, $d>0.01$, the behavior of $J(\nu)$ is as follows. If $a_0>a_1(D)$, then two CRs occur as the correlation time grows from 0 to ∞. For increasing values of ν, the current starts from a positive value determined by Eq. (26), decreasing to a negative local minimum, next it grows, attaining a positive (very small in most cases) local maximum, and then J approaches zero as $\nu\to\infty$. The local maximum of the current J increases and the corresponding switching rate ν_m $[J_{max}=J(\nu_m)]$ decreases monotonically as the temperature increases. The current changes its sign at the two noise correlation time values $\tau_1=1/\nu_1$ and $\tau_2=1/\nu_2$. A growth of the temperature causes the bigger of the two solutions of $J(\nu)=0$, ν_2, to drop monotonically from infinity to a certain finite value, while the other solution ν_1 is nonmonotonic and has a minimum at a finite value of $D=D_1$. It should be noted that the temperature D_m at which the absolute value of the local minimum of the current is maximal does not differ much from the temperature D_1, exceeding it only slightly.

At low temperatures, $D<D_1$, one has to discern two cases. First, if $a_0<bc$, then ν_1 tends to infinity if D tends to zero. Second, if $a_0>bc$, then at decreasing D the parameter ν_1 approaches a finite value $\nu_1(0)$ that can be found from the transcendental equation $[\nu_0=\nu_1(0)]$

$$e^{\nu_0/c(a_0-c)}+e^{-\nu_0/c(a_0+c)}=e^{\nu_0/b(a_0-b)}+e^{-\nu_0/b(a_0+b)}. \tag{39}$$

Notably, at large values of the amplitude, $a_0\gg bc$, the minimum $\nu_1(D_1)$ of ν_1 is practically equal to the parameter ν_0, i.e., $\nu_1(D_1)\approx\nu_1(0)$.

If $D>D_1$, then ν_1 increases as D grows: there is another characteristic temperature $D_c>D_1$ at which the phenomenon of CR disappears. For $D=D_c$, the parameters ν_1 and ν_2 coincide $(\nu_1=\nu_2)$ and $a_1(D_c)=a_0$. If $D>D_c$, then the current is positive. The corresponding typical graphs of $J(\nu)$ are given in Fig. 5 as the curves (1) and (2). For high temperatures, where $D\gg D_c$, the curve $J(\nu)$ is always monotonic.

According to numerical calculations for $d>0.001$ we can notice that at sufficiently large values of the noise amplitude, the critical temperature D_c is proportional to the amplitude. At moderate asymmetries of the potential V $(d>0.2)$, the factor of proportionality takes a remarkably simple form:

$$D_c\approx\frac{a_0}{\sqrt{3}bc},\quad a_0>1. \tag{40}$$

The tendency that is apparent at Eq. (40), namely a decrease of the critical temperature as the asymmetry of the potential grows, is also valid for large asymmetries $(d<0.2)$.

At some cases, one can find rather simple expressions for the approximation of the parameter ν_1. For example, in the case of a large noise amplitude $a_0\gg\max\{bc,\sqrt{3}bcD\},d>0.2$, the value of the first reversal point ν_1 can be estimated by the following equation:

$$\nu_1\approx\frac{\nu_1(0)}{f(D)}, \tag{41}$$

where $f(D)$ is given by Eq. (21). Specifically, for $D\ll1$ we have $\nu_1\approx\nu_1(0)(1+4D)$, and if $D\gg1$, then $\nu_1\approx180\nu_1(0)D^3$.

Notably, at large potential asymmetries, $d<0.01$, we have noticed remarkable exceptions from the described behavior of $J(\nu)$. For instance, if $d=0.005$ and $a_0=23$, there can be four CRs with five critical temperatures: $D_{c1}\approx0.334$, $D_{c2}\approx0.280$, $D_{c3}\approx0.062$, $D_{c4}\approx0.058$, $D_{c5}\approx0.050$. Within the values of $D>D_{c1}$ and $D_{c3}<D<D_{c2}$, there is no CR; within $D_{c2}<D<D_{c1}$, $D_{c4}<D<D_{c3}$, and $D<D_{c5}$, there are two CRs; within $D_{c5}<D<D_{c4}$, there are just four CRs.

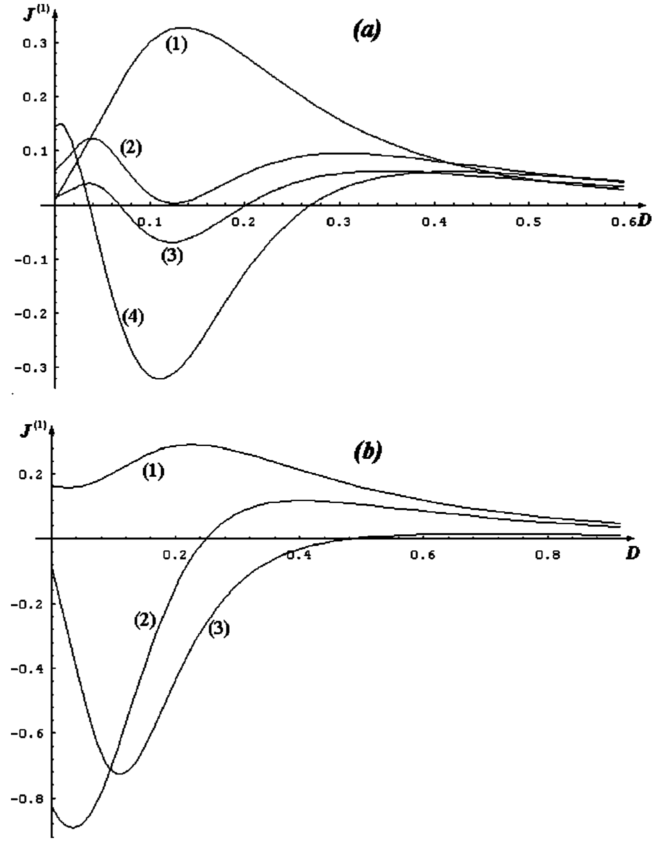

FIG. 6. The current $J=qJ^{(1)}$ vs the temperature D in the case of large flatness [Eq. (31)] at $d=0.25$. In the limit of high temperature, J is positive and decays to zero in D^{-4}. (a) The case of $a_0<bc$. Curves (1)–(4) correspond to the following parameters: (1) $a_0=1.5$, $\nu=1$; (2)$a_0=3.5$, $\nu=16$; (3) $a_0=3.5$, $\nu=21$; (4) $a_0=4.5$, $\nu=21$. If $\nu>\nu_1(D_1)$, then two current reversals occur [curves (3) and (4)]. The parameter $\nu_1(D_1)$ can be found by using Eq. (31). (b) The case of $a_0>bc$. Here $a_0=6$ and the curves correspond to (1) $\nu=3$, (2) $\nu=10$, and (3) $\nu=50$. If $\nu>\nu_0\approx3.751$, one current reversal appears [curves (2) and (3)]. The critical switching rate ν_0 is the solution of Eq. (39). No current reversals occur when $\nu<\nu_1(D_1)\approx3.735$ [curve (1)].

E. Current versus temperature

In Fig. 6, we have plotted the current $J(D)$ as a function of the temperature D. It can be seen that the effect of CR can be attained also by changing the temperature D. In a general case, the dependence of $J(D)$ is nonmonotonic and there are one to three characteristic (optimal) temperatures at which the left and right currents are maximized. It is remarkable that there exist both the critical value $a_c=bc$ of the amplitude a_0 and the critical value $\tau_1(a_0)=1/\nu_1(D_1)$ of the correlation time τ_c.

(i) If $\tau_c>\tau_1(a_0)$, then there is no CR. The current is positive at all temperatures. The concrete value of $\tau_1(a_0)$ for a given a_0 and asymmetry of the potential can be found by numerical methods from Eq. (31).

(ii) In the cases of large amplitudes $a_0 > bc$, another critical value of the correlation time $\tau_2(a_0) = 1/\nu_0 < \tau_1(a_0)$ appears, where ν_0 is the solution of Eq. (39). If $a_0 > bc$ and $\tau_2 < \tau_c < \tau_1$, there are two CRs. But if $a_0 > bc$ and $\tau_c < \tau_2$, there is only one CR. Notably, at $a_0 > bc$ the phenomenon of two CRs would be hardly noticeable, for τ_2 and τ_1 can differ materially from each other only if $a_0 \approx bc$. For example, if $d = 0.25$ ($a_c = 16/3$), then it follows that (a) if $a_0 = 5.4$, then $D_1 \approx 0.08$, $\tau_1 \approx 0.167$, $\tau_2 \approx 0.093$; and (b) if $a_0 = 6$, then $D_1 \approx 0.016$, $\tau_1 \approx 0.268, \tau_2 \approx 0.267$. A further growth of the amplitude would cause a quick drop of the difference of τ_1 and τ_2.

(iii) In the case of $a_0 < bc$ and $\tau_c < \tau_1(a_0)$, there occur either two CRs or none. For moderate asymmetries of the potential, $d > 0.01$, we have always seen two CRs. At large asymmetries, $d < 0.01$, CRs can be absent in the region $c \ll a_0 \ll b$ and there can be several critical correlation times; for example, there are three in the case of $d = 0.005$ and $a_0 = 23$, namely $\tau_1 \approx 6.00 \times 10^{-3}$, $\tau_2 \approx 4.57 \times 10^{-3}$, and $\tau_3 \approx 1.34 \times 10^{-3}$. In the regions of $\tau_c > \tau_1$, $\tau_3 < \tau_c < \tau_2$, there is no CR, and in $\tau_2 < \tau_c < \tau_1$, $\tau_c < \tau_3$, there are two CRs.

The characteristic temperatures D^* at which the current is zero are sensitive to variation of the model parameters a_0, d, and ν. Note that in Eq. (3) the friction coefficient of the particle has been absorbed into the time scale. Thus, in the original (unscaled) setup, particles with different friction coefficients are controlled by different effective ν's and can move in either direction in the same ratchet potential and the same fluctuating environment, which has interesting biological and technological implications (see [1–3,6,7,9]).

V. THE CASE OF A SMALL NOISE AMPLITUDE

Here we consider the case of $a_0 \ll \min\{c,D\}$. We can start with the ansatz

$$P(x) = \frac{e^{-V(x)/D} e^{1/D}}{D(e^{1/D} - 1)} + a_0^2 \tilde{P}(x)$$

in Eq. (8). Collecting the terms of the order a_0^2, we get an inhomogeneous second-order equation for $\tilde{P}(x)$. In the case of a sawtoothlike potential (12), this equation is immediately solvable. As the corresponding expression for the current is rather cumbersome, we would rather not present it here. However, the current J can be given the form

$$J = a_0^2 \lim_{a_0^2 \to 0} \frac{qJ^{(1)}}{a_0^2}, \qquad (42)$$

where $qJ^{(1)}$ is determined by Eqs. (31)–(33). It should be noted that Eq. (42) is valid for all values of the flatness parameter $\varphi = 1/2q \geq 1$.

The net current is positive and exhibits a bell-shaped extremum (see Fig. 7). As the correlation time increases, J decreases monotonically from

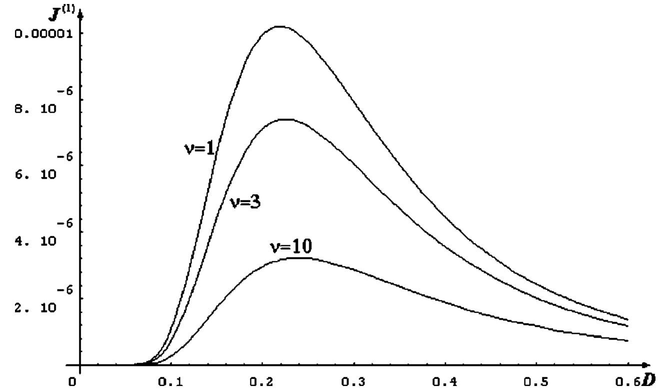

FIG. 7. The current $J = qJ^{(1)}$ vs the temperature D for a varying switching rate ν in the case of a small noise amplitude $a_0 = 0.01$ and $d = 0.25$ [Eq. (42)]. The current is positive and exhibits a bell-shaped form for all values of the flatness parameter φ.

$$J = \frac{q a_0^2 (b-c) e^{1/D}}{D^3 bc (e^{1/D} - 1)^2} f(D)$$

to zero [see also Eq. (28)]. A direct numerical calculation of the optimal temperature D_m, at which the current is maximized, indicates that D_m is a slowly varying function of the correlation time τ_c and of the asymmetry of the potential d (see Fig. 8). For all values of the parameters d and τ_c, the optimal temperature D_m is in the interval (0.21,0.32). As the correlation time decreases, the optimal temperature increases from $D_m \approx 0.216$ to a local maximum (that is small and hardly discernible) and then decreases to 0.309. This is a remarkable result, for it indicates that there is some robustness in the system: it can be assumed that the interval (0.21,0.32) of the temperature is optimal for a large class of overdamped ratchet models with an additive thermal noise and with an additive low-amplitude nonequilibrium noise.

VI. CONCLUDING REMARKS

Above, we have presented some analytical and exact results for the dynamics of an overdamped Brownian particle in a sawtooth ratchet potential subjected to both thermal noise and zero-mean exponentially correlated three-level fluctuations (trichotomous noise). A major virtue of the models with trichotomous noise is that they constitute another case admitting an exact analytical solution for the stationary current for any value of the correlation time $\tau_c = 1/\nu$, the noise amplitude a_0, and the flatness parameter φ.

For both slow and fast fluctuating forces, we have presented approximations that agree with the results of Refs. [4,15,16,24]. In the case of large noise amplitudes ($a_0 \to \infty$) with the other parameters fixed, the current saturates to a finite negative value [Eq. (17)]. This result is due to the so-called flashing barrier effect as stated in Refs. [20,23]. It is remarkable that in the cases of the adiabatic limit, fast-noise limit, small-amplitude limit, and high-temperature limit, the current takes the same qualitative form as in the corresponding cases with a dichotomous noise. The presence of flatness $\varphi > 1$ modifies the prefactor in a simple way: the

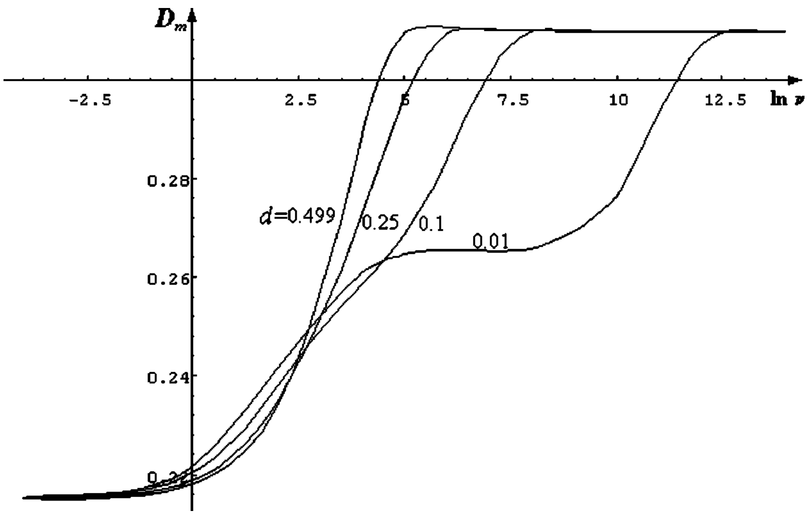

FIG. 8. The temperature D_m that maximizes the current in the case of small-noise amplitude vs $\ln\nu$ for some values of the potential asymmetry parameter d. The optimal temperature D_m does not depend on the flatness parameter and lies in the interval (0.21,0.32) at all values of d, and ν. We can see that at small values of d, an additional maximum and minimum of D_m appear.

current is proportional to $1/\varphi$. This indicates that in the cases mentioned above, the flashing barrier effect can be neglected.

The thermal noise had a profound effect on the magnitude of the current in the small τ_c limit. By an absence of thermal noise, the flatness parameter φ has a critical value $\varphi=2$ for the greater values of which at sufficiently large values of the amplitude a_0, the current $J(\tau_c)$ versus correlation time can be characterized by one sign reversal of J (CR); in the small τ_c limit, the current is transcendentally small and negative. In the presence of additional thermal noise, there can be either an even number of CRs or none; in the small τ_c limit, J is positive and decays algebraically in $\tau_c^{5/2}$. Notably, in the case of a symmetric dichotomous noise ($\varphi=1$), there is no CR and the current is positive at all temperatures.

An interesting circumstance concerning the ratchet models with a trichotomous noise is that for some system parameters there occur more than two CRs. As far as we know, more than two CRs have never been seen for correlation ratchets so far. At the same time, in the case of inertial deterministic rocking ratchets, the current can exhibit infinitely many reversals [33,34].

Our major result is that in sawtooth ratchet structures, the direction of the transport of Brownian particles driven by trichotomous fluctuations can be controlled by thermal noise (see Fig. 6). The necessary condition is that the flatness parameter exceeds 1. The advantage of this model is that the control parameter is temperature, which can easily be varied in experiments (see also [17]). The discovery of temperature regions in which particles of different friction coefficients are transported in opposite directions could be implemented in an effective method of particle separation as suggested in [8,17,21,22,35]. For large flatnesses ($\varphi \gg 1$), we have elaborated on the conditions for the noise parameters and temperature leading to a sign reversal of J. If the control parameter is temperature, then there is an upper limit $\tau_1(a_0)$ for the correlation time τ_c, at greater values of which there is no CR. For $a_0 > bc$, another critical value of the correlation time $\tau_2(a_0) < \tau_1(a_0)$ occurs. In the case $\tau_2 < \tau_c < \tau_1$ there are two CRs, but at $\tau_c < \tau_2$ there is only one CR.

For $a_0 < bc$ and $\tau_c < \tau_1(a_0)$, two or no CRs appear. They can be absent only at the values of the noise amplitude $c \ll a_0 \ll b$, which is possible only at large asymmetries of the potential $d < 0.01$. Then the current exhibits disjunct characteristic ''windows'' of the correlation time where the temperature-controlled CRs take place.

In the case of a small noise amplitude, $a_0 \ll \min\{c, k_B T/\tilde{V}_0\}$, where $\tilde{V}_0$ is a barrier height of the periodic potential $\tilde{V}(x)$, the current is positive, and it exhibits a bell-shaped extremum (see Fig. 7). The temperature T_m at which J is maximized weakly depends on the other parameters (φ, d, and τ_c) and always lies in the interval $(0.21\tilde{V}_0/k_B, 0.32\tilde{V}_0/k_B)$. It seems reasonable to assume that this temperature interval is optimal for a large class of overdamped ratchet models with additive thermal noise and an additive low-amplitude nonequilibrium noise. It remains to be seen whether such a tolerance of T_m can play a role in the problems of natural sciences.

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